Indirect application of a method for discretization of linear fractional order systems

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Abstract—**A method for discretization of linear fractional order systems (LFOS) is presented. Basically, the method makes use of rational approximation of the transfer function of a fractional system. In this way standard discretization techniques for discretization of LFOS can be applied. An adequate comparative analysis has also been carried out through corresponding examples by applying several other known discretization methods.**

Keywords - discretization; fractional order systems; rational approximations

I. INTRODUCTION

Designing classical and/or fractional order control laws involving integral and differential actions [1,2] often requires formulation of a discrete model of the process by using methods of invariable response to a pulse or Heaviside excitation and a series of other approximate methods [3-20]. Since a process can, in general, be represented by a transfer function $G_p(s)$ which is not a rational function but which describes a fractional order system [21,22], the problem of disretization becomes complex. In addition, the basic problems required to be solved by the process of discretization of such, or similar, control systems are related to retaining the fundamental system properties, such are steady-state gain and settling time, as well as basic properties in the frequency domain. In the process of discretization of LFOS, where, in general, fractional order integral and differential actions belong, one can make use of the well-known mapping of s-domain to z-domain in the complex plane

$$z = e^{sT}, \tag{1}$$

where T is the sampling time. Transform (1) maps left half-plane of the s-plane to interior of the unity circle in the z-plane. This means that stability of the discrete system has been preserved if all poles of the discrete system are located within the unity circle. One of the basic goals of discretization is acquiring the ability for practical realization of the corresponding control laws or of some other requirements in order that the digital model is fully equivalent to the continuous system over a wide frequency range.

For the purpose of illustration of LFOS, consider a process described by classical diffusion equation (also referred to as the heat equation), which is ubiquitous in science and engineering since it simultaneously describes a number of transfer phenomena, including heat-transfer and a number of other diffusion-like processes. These diffusion-like processes include diffusion of mass (mechanical diffusion), diffusion of momentum (viscosity), diffusion of electrical potential (in long lines, when inductivity is negligible), and many others. One-dimensional diffusion equation is a partial differential equation of the form

$$\tau \frac{\partial^2 \rho}{\partial z^2} = \frac{\partial \rho}{\partial t}, \quad \tau > 0 \tag{2}$$

describing the process of transport (diffusion) of a quantity ρ along the z axis in time t. For simplicity, let us address only the diffusion within a semi-infinite medium, where both space and time variable take arbitrary positive values. Let us assume also that the process can be controlled by acting on the cross-section $z = 0$, and that the process output is taken (measured) at the cross-section $z = L$. The dynamics of the process is influenced by the diffusion time constant $\tau = \tau(z,t)$, which is, in general, a function of both space and time. However, in a variety of practically interesting cases this coefficient can be approximated by a constant factor.

Without loss of generality, assume that (2) describes a heat conduction process schematically shown in Fig. 1. Let us obtain its transfer function. In this particular case, $\rho = \rho(z,t)$, is the temperature of the cross section defined by space coordinate z evaluated at time instant t. Let $\tilde{\rho} = \tilde{\rho}(z,s)$ denote the Laplace transform of ρ, where the Laplace transform is taken with respect to the time variable t and the space variable z is considered as a parameter,

$$\tilde{\rho}(z,s) = \int_0^\infty \rho(z,t) e^{-st} dt \,. \tag{3}$$

By applying the Laplace transform to equation (2), one obtains general solution

$$\tilde{\rho}(s,z) = C_1(s)\, e^{-z\sqrt{s/\tau}} + C_2(s)\, e^{z\sqrt{s/\tau}} \,. \tag{4}$$

Since any heat conduction process is stable, the Laplace transform of the temperature in any cross-section must be bounded, i.e.

$$\lim_{z\to\infty}\tilde{\rho}(s,z)=const. \tag{5}$$

has to be satisfied, thus $C_2=0$ and equation (4) takes the form

$$\tilde{\rho}(s,z)=C_1(s)\,e^{-z\sqrt{s/\tau}}\,. \tag{6}$$

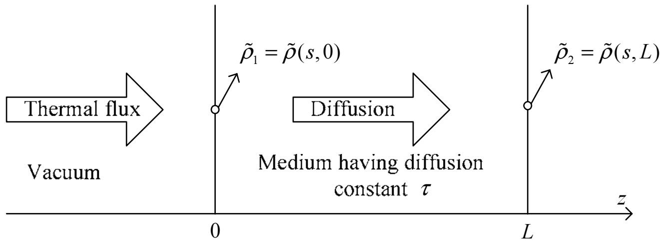

Figure 1. A sketch of the process of heat conduction by diffusion.

Integration "constant" C_1 as well as the conduction function is determined from the known (or given) boundary conditions. In view of this, the most frequent cases in practice are:

Case 1. Heat conduction without any convective exchange of heat with the environment and fixed temperature at the "left" boundary. In this particular case, the temperature of the cross-section $z=0$ could be controlled directly, and considered as the input of the process, while the dependent temperature of the cross-section $z=L$ could be considered as the output. The left boundary condition for this case is $\tilde{\rho}(s,0)=C_1(s)$, and the transfer function takes the form

$$G_a(s)=\frac{\tilde{\rho}_2(s,L)}{\tilde{\rho}_1(s,0)}=e^{-L\sqrt{s/\tau}}=e^{-\sqrt{Ts}}\,,\ T=L^2/\tau\,. \tag{7}$$

Case 2. Heat conduction without any convective exchange of heat with the environment and fixed thermal flux at the "left" boundary. The process is influenced by gradient of quantity ρ at z = 0 (this is the boundary surface of the medium of Fig. 1), the input quantity of the process being thermal flux through the boundary surface (again without any convective exchange with the environment)

$$\psi=-\lambda\left.\frac{d\tilde{\rho}(s,z)}{dz}\right|_{z=0} \tag{8}$$

and the process (output) quantity is $\rho_2=\rho(s, \mathrm{L})$, and the transfer function is

$$G_b(s)=\frac{\tilde{\rho}_2}{\psi}=\frac{K}{\sqrt{s}}e^{-\sqrt{Ts}}\,,\ T=L^2/\tau\,,\ K=\sqrt{\tau}/\lambda\,. \tag{9}$$

Case 3. Heat conduction without any convective exchange of heat with the environment. The last characteristic case is when the convection is no longer neglected. Now, the process is influenced by a linear combination of the thermal flux and temperature at the "left" boundary

$$u=-\lambda\left.\frac{d\tilde{\rho}(s,z)}{dz}\right|_{z=0}+\eta\,\tilde{\rho}(s,z)\,, \tag{10}$$

with output $\rho_2=\rho(s,L)$, and the transfer function is

$$G_c(s)=\frac{\tilde{\rho}_2}{u}=\frac{K}{1+\sqrt{T_1 s}}e^{-\sqrt{Ts}}\,,\ K=\frac{1}{\eta},\ T_1=\frac{\lambda^2}{\eta^2\tau},\ T=\frac{L^2}{\tau}\,. \tag{11}$$

In the examples above, the semi-derivative operator has appeared in a number of contexts. It should be mentioned that other forms of fractional order transfer functions emerge during investigations of different transfer phenomena. In the analysis of axial diffusion, i.e. diffusion from the axis of the cylinder towards its lateral surface or vice versa, one meets transfer functions originating from the Laplace transforms of Bessel functions, which have the form

$$G(s)=\frac{K}{\sqrt{1+sT}} \tag{12}$$

From this example, transfer functions given by equations (7), (9), (11), and (12) belong to the fractional order systems having transfer functions which belong to the class of irrational functions [23,24].

Since these transfer functions describe adequately physical processes, a logical question arises whether it is possible to formulate fractional order control laws and what would be their contribution to process control. Among many modern control strategies utilizing fractional order calculus, Podlubny's Fractional order PID [1,2] regulator is emphasized here. Classical PID is arguably the most utilized control strategy in use today. By replacing classical integral and differential actions by their respective fractional order analogues, the flexibility and applicability of the PID regulator can be greatly increased. Transfer function of the fractional order PID is of the form

$$\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}(s)=k+k_{\mathrm{i}}s^{-\lambda}+k_{\mathrm{d}}s^{\mu}\,,\ \lambda,\mu\in[0,1]\,. \tag{13}$$

The reader should notice that the implementation of Fractional order PID requires direct implementation of fractional order integrator and differentiator. Similar is also true for other

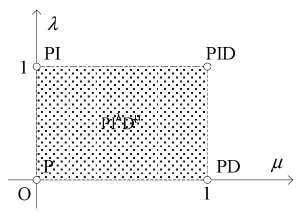

Figure 2. Parameter plane of the fractional order $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ regulator

types of fractional order regulators, as it can be seen from [25-27]. Such regulators are typically implemented as high order FIR or IIR filters [28], Realization of fractional order control laws involving an adequate discretization is possible thanks to the fast modern computers. It is known that in the regulator design two approaches are possible, direct design in the discrete domain and the other approach is design in the continuous domain first and then transition to the discrete domain. Obviously, discretization is required by both approaches. However, the discretization procedure is not straightforward when fractional order systems are in question, a problem which has been causing a considerable interest over the past years.

The paper comprises introduction, main part, conclusions, and references. In the first portion of the main part the method of rational approximation of the transfer function of LFOS is presented. Then, a comparative analysis of the responses to Heaviside excitation of the continuous and digital equivalent of EFS and their frequency characteristics within a specified frequency range is given.

II. Rational Approximation of Transfer Function of LFOS

Let us consider rational transfer function

$$\frac{B(s)}{A(s)} = \frac{b_{n-1}s^{n-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \qquad (14)$$

which should be used to approximate transfer function $G(s)$ of a linear fractional order system. For ($G(0)\neq 0$, $b_0=1$) or ($G(0)=0$, $a_0=1$) there are 2n real coefficients which should be determined from 2n equations obtained from the condition of overlapping the frequency characteristics in the corresponding discrete frequency points $\omega \in [\omega_0, \omega_2, \cdots, \omega_{n-1}]$, i.e.

$$G(\mathrm{i}\omega_k) - B(\mathrm{i}\omega_k)/A(\mathrm{i}\omega_k) = 0,\ k = \overline{0, n-1},\ \mathrm{i} = \sqrt{-1}, \qquad (15)$$

or for $G(0)\neq 0$, $b_0=1$ one obtains

$$\mathrm{Re}\left(A(\mathrm{i}\omega_k) - B(\mathrm{i}\omega_k)/G(\mathrm{i}\omega_k)\right) = 0,\ k = \overline{0, n-1}, \qquad (16)$$

$$\mathrm{Im}\left(A(\mathrm{i}\omega_k) - B(\mathrm{i}\omega_k)/G(\mathrm{i}\omega_k)\right) = 0,\ k = \overline{0, n-1}. \qquad (17)$$

Set of equations (16), (17) represents a linear system of equations having 2n unknown coefficients. By solving this system of 2n linear equations, one obtains 2n coefficients of rational approximation (14).

It is important to mention that the selected set of points $\omega \in [\omega_0, \omega_2, \cdots, \omega_{n-1}]$ can produce a singular matrix of the set of equations (16), (17) which, in this case, should be solved by using pseudo-inverse matrix methods or another set of points should be applied which results in a regular system matrix of the set of equations (16), (17). It is also significant to note that it is also possible to use more than n incident points in the selected set. The exact solution cannot be found in such a case. However, the best approximation, in the least-squares sense, can be found by means of pseudo-inversion.

A. Simulation analysis

Let us select several LFOS transfer functions and compare their Bode characteristics and responses to Heaviside excitation with those of the corresponding rational approximations determined on the basis of the set of equations (16), (17).

Example 1. $G_1(s) = 1/(s^{3/2}+1)$, $\omega \in [0.01, 0.1, 0.5, 1, 5, 10, 100]$

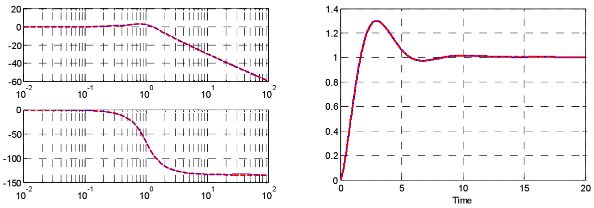

Figure 3. Frequency and time responses $G_1(s)$ (red) i $B_1(s)/A_1(s)$ (blue)

Example 2. $G_2(s) = 1/(s - \sqrt{2s} + 1)$, $\omega \in [0.01, 0.1, 0.5, 1, 5, 10, 100]$

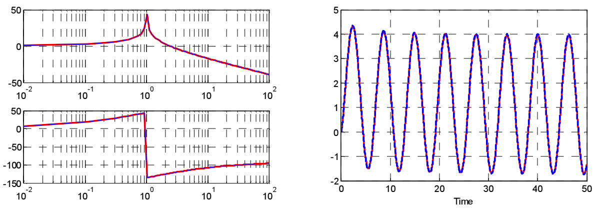

Figure 4. Frequency and time responses $G_2(s)$ (red) i $B_2(s)/A_2(s)$ (blue)

Example 3. $G_3(s) = \exp(-\sqrt{s})$, $\omega \in [0.01, 0.1, 1, 2, 10, 50, 100]$.

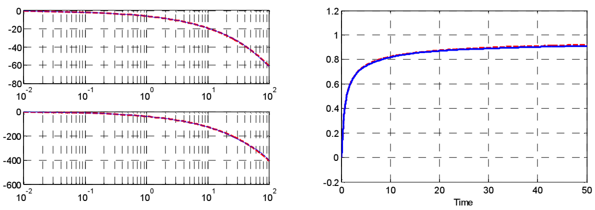

Figure 5. Frequency and time responses $G_3(s)$ (red) i $B_3(s)/A_3(s)$ (blue)

Example 4. $G_4(s) = \ln(s)/s$, $\omega \in [0.001, 0.01, 0.1, 0.5, 1, 5, 50]$.

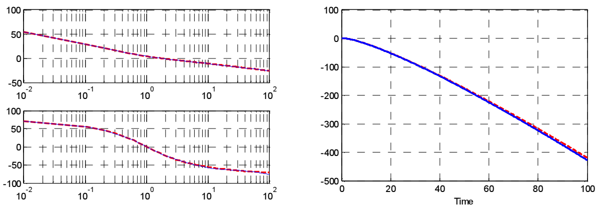

Figure 6. Frequency and time responses $G_3(s)$ (red) i $B_3(s)/A_3(s)$ (blue)

Example 5. $G_5(s) = \left(\dfrac{s+1}{0.1s+1}\right)^{0.5}$, $\omega \in [0.01, 0.1, 1, 3, 5, 10, 100]$.

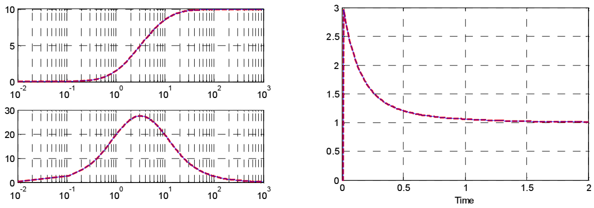

Figure 7. Frequency and time responses $G_5(s)$ (red) i $B_5(s)/A_5(s)$ (blue)

Example 6. $G_6(s) = (1 + 1/s + s^{1.2})/(0.1s+1)^{1.2}$,

$\omega \in [0.5, 0.8, 1, 2, 5, 30, 100]$

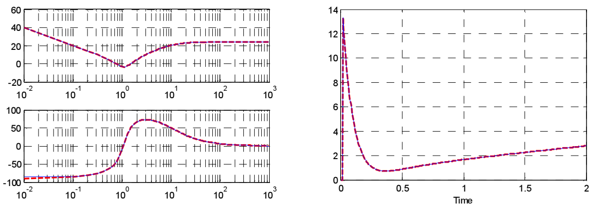

Figure 8. Frequency and time responses $G_6(s)$ (red) i $B_6(s)/A_6(s)$ (blue)

As the previous figures show, the rational approximations give adequate approximations for a wide range of LFOS. It should especially be mentioned that the corresponding frequency points are selected on the basis of the knowledge of dynamics of Bode characteristics of LFOS transfer functions. In all previous examples the selected order $n = 7$ obviously can be lower, but under condition that the frequency characteristics in the selected frequency range are not violated. Also, for the selected set of frequencies in all these examples, the matrices of the sets of equations (16), (17) have been regular.

Obviously now, since a rational transfer function is in question, applications of all techniques of discretization are possible, consequently adequate discrete models of LSF transfer functions within certain frequency range are available.

All of the time-domain responses presented above are obtained by means of direct integration in the complex domain. The interested reader is referred to [14].

III. Conclusions

Owing to simplicity of application of the method of rational approximation of transfer functions of linear fractional order systems, this paper is dedicated to an analysis of the application of this approach for the purpose of discretization of linear fractional order systems. It should be emphasized that, since a rational transfer function of a continuous system is in question, application of all techniques of its discretization are possible, i.e. for discretization of linear fractional systems.

Further investigations should tackle the problem of finding an optimum degree of rational approximation within a selected frequency range and the method of selecting frequency points within this range in order to obtain as good approximation as possible. Also, future research will address the problem of investigating (and possibly guaranteeing) the stability of the obtained approximations, under the assumption that the original fractioanal order system is itself stable.

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